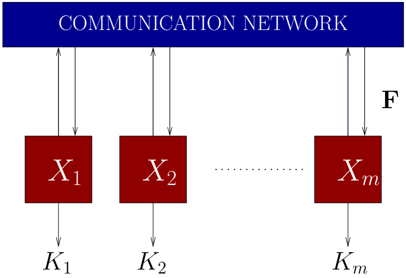
A Bound For Multiparty Secret Key Agreement

And

Implications For A Problem Of Secure Computing

Himanshu Tyagi and Shun Watanabe

Multiparty Secret Key Agreement



Party i computes $K_i(X_i, \mathbf{F}) \in \mathcal{K}$; Eavesdropper observes $\mathbf{F}, Z$

$K_1, ..., K_m$ constitute an (ϵ, δ)-secret key of length $\log \mathcal{K}$ if

$$\mathrm{P}\left(K_1 = K_2 = ... = K_m\right) \geq 1 - \epsilon, \qquad \text{:Recoverability}$$

$$\frac{1}{2}\|\mathrm{P}_{K_1\mathbf{F}Z} - \mathrm{P}_{\mathtt{unif}} \times \mathrm{P}_{\mathbf{F}Z}\|_1 \leq \delta, \qquad \text{:Secrecy}$$

Alternative Definition of a Secret Key

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Lemma

(ϵ, δ)-SK$\Rightarrow$ $(\epsilon + \delta)$-SK, and conversely, ϵ-SK $\Rightarrow$ (ϵ, ϵ)-SK.

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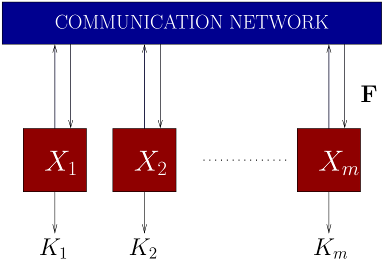

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Definition

$S_\epsilon(X_1, \ldots, X_m \mid Z) \triangleq$ maximum length of an ϵ-secret key

Upper bound for $S_\epsilon(X_1, ..., X_m \mid Z)$

No Correlation No Secret Key

If X_1 and X_2 are independent conditioned on Z:

$$S_\epsilon(X_1, X_2|Z) \approx 0$$

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Bound $S_\epsilon(X_1, ..., X_m|Z)$ in terms of "how far" is $\mathrm{P}_{X_1,...,X_mZ}$ is from a conditionally independent distribution

Digression: Binary Hypothesis Testing

Consider the following binary hypothesis testing problem:

$$\begin{aligned} H0: &\quad X \sim P \\ &vs. \\ H1: &\quad X \sim Q \end{aligned}$$

Define

$$\beta_\epsilon(P, Q) \triangleq \inf \sum_{x \in \mathcal{X}} Q(x) T(0|x),$$

where the $\inf$ is over all random tests $T : \mathcal{X} \to \{0, 1\}$ s.t.

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Data processing. For every stochastic matrix $W : \mathcal{X} \to \mathcal{Y}$

$$\beta_\epsilon(P,Q) \le \beta_\epsilon(PW, QW)$$

Reduction Argument

Given a partition $\pi = \{\pi_1, ..., \pi_k\}$ of $\{1, ..., m\}$

- Let $\mathrm{Q}(x_1, ..., x_m|z) = \prod_{i=1}^{k} \mathrm{Q}(x_{\pi_i}|z)$

For the binary hypothesis testing:

$$\begin{aligned} H0: &\quad X_1, ..., X_m, Z \sim \mathrm{P}, \\ H1: &\quad X_1, ..., X_m, Z \sim \mathrm{Q}, \end{aligned}$$

consider the degraded observations $K_1, ..., K_m, \mathbf{F}, Z$.

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Let $W_{K_1 \dots K_m \mathbf{F}|X_1 \dots X_m Z}$ represent the protocol.

Reduction Argument

Consider the degraded binary hypothesis testing:

$$\begin{aligned} H0: &\quad K_1, ..., K_m, \mathbf{F}, Z \sim \mathrm{P}_{K_1 ..., K_m \mathbf{F} Z} = \mathrm{P}W \\ H1: &\quad K_1, ..., K_m, \mathbf{F}, Z \sim \mathrm{Q}_{K_1 ..., K_m \mathbf{F} Z} = \mathrm{Q}W \end{aligned}$$

Consider a test with the acceptance region $\mathcal{A}$ defined by:

$$\mathcal{A} \triangleq \left\{ \log \frac{\mathrm{P}_{\mathrm{unif},m}(K_1, ..., K_m)}{\mathrm{Q}_{K_1 ... K_m | \mathbf{F} Z}(K_1 ... K_m | \mathbf{F}, Z)} \geq \lambda_\pi \right\}$$

where

$$\lambda_\pi = (|\pi| - 1) \log |\mathcal{K}| - |\pi| \log(1/\eta)$$

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Likelihood ratio test with $\mathrm{P}_{K_1 ... K_m | \mathbf{F} Z}$ replaced by $\mathrm{P}_{\mathrm{unif},m}$

- recall: $\frac{1}{2} \| \mathrm{P}_{K_1 K_2 ... K_m \mathbf{F} Z} - \mathrm{P}_{\mathtt{unif},m} \times \mathrm{P}_{\mathbf{F} Z} \|_1 \leq \epsilon$

Reduction Argument

Missed Detection: $\mathrm{Q}_{K_1 \dots K_m \mathbf{F} Z}(\mathcal{A}) \leq |\mathcal{K}|^{1-|\pi|} \eta^{-|\pi|}$

False Alarm: $\mathrm{P}_{K_1 \dots K_m \mathbf{F} Z}(\mathcal{A}^c) \leq \epsilon + \eta$

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Lemma (Reduction)

For every $0 \leq \epsilon < 1$ *and* $0 < \eta < 1 - \epsilon$,

$$S_\epsilon(X_1, \ldots, X_m|Z) \leq \frac{1}{|\pi| - 1} \left[-\log \beta_{\epsilon+\eta} \left(\mathrm{P}W, \mathrm{Q}W \right) + |\pi| \log \left(1/\eta \right) \right].$$

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By data processing: $\beta_{\epsilon+\eta}\left(\mathrm{P}W, \mathrm{Q}W\right) \geq \beta_{\epsilon+\eta}\left(\mathrm{P}, \mathrm{Q}\right)$

Conditional Independence Testing Bound

Theorem

For every $0 \leq \epsilon < 1$ *and* $0 < \eta < 1 - \epsilon$,

$$S_\epsilon(X_1, ..., X_m|Z) \leq \frac{1}{|\pi| - 1} \left[- \log \beta_{\epsilon+\eta} \left(\mathrm{P}, \mathrm{Q} \right) + |\pi| \log \left(1/\eta \right) \right],$$

where

$$\mathrm{Q}(x_1, ..., x_m|z) = \prod_{i=1}^{k} \mathrm{Q}(x_{\pi_i}|z).$$

For two parties:

$$S_\epsilon(X_1, X_2|Z) \leq - \log \beta_{\epsilon+\eta} \left(\mathrm{P}_{X_1 X_2 Z}, \mathrm{P}_{X_1|Z} \mathrm{P}_{X_2|Z} \mathrm{P}_Z \right) + 2 \log \left(1/\eta \right)$$

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Connections to meta-converse of Polyanskiy, Poor, and Vérdu

Implications of the Upper Bound

1. Strong Converse for Secret Key Agreement

[Maurer '93] [Ahlswede-Csiszár '93] [Csiszar-Narayan '04]

Consider IID observations $X_1, ..., X_m \equiv X_1^n, ..., X_m^n$, $Z = \emptyset$

(ϵ, δ)-Secret Key Capacity: $C_{\epsilon,\delta} := \liminf_n \frac{1}{n} S_{\epsilon,\delta}(X_1^n, ..., X_m^n)$

Secret Key Capacity: $C := \inf_{\epsilon,\delta} C_{\epsilon,\delta}.$

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Theorem

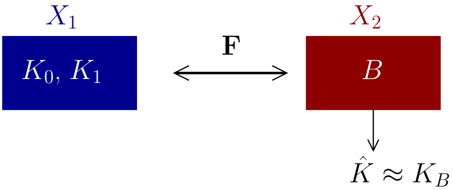
For $0 < \epsilon, \delta$ *with* $\epsilon + \delta < 1$,

$$C_{\epsilon,\delta} = C,$$

and for all $\epsilon + \delta \geq 1$,

$$C_{\epsilon,\delta} = \infty.$$

2. Information Theoretically Secure OT



[Even-Goldreich-Lempel 85], ..., [Nascimento-Winters 06]

- Reliability: $\mathrm{P}\left(\hat{K} \neq K_B\right) \leq \epsilon$
- Security 1: $\frac{1}{2}\left\|\mathrm{P}_{BK_0K_1X_1\mathbf{F}} - \mathrm{P}_B \times \mathrm{P}_{K_0K_1X_1\mathbf{F}}\right\|_1 \leq \delta_1$
- Security 2: $\frac{1}{2}\left\|\mathrm{P}_{K_{\overline{B}}BX_2\mathbf{F}} - \mathrm{P}_{K_{\overline{B}}} \times \mathrm{P}_{BX_2\mathbf{F}}\right\|_1 \leq \delta_2$

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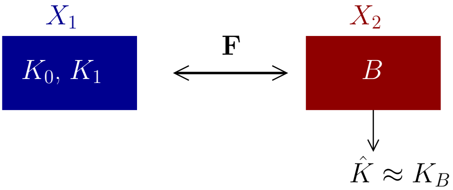

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How large can the length l of OT be?

Bounds on the Efficiency of OT

Theorem (Reduction of SK Agreement to OT)

For an $(\epsilon, \delta_1, \delta_2)$*-OT of length* l

$$l \lesssim \min\left\{S_{\epsilon+\delta_1+2\delta_2}(X_1, X_2),\ S_{\epsilon+\delta_1+2\delta_2}\left(X_1, (X_1, X_2) \mid X_2\right)\right\}$$

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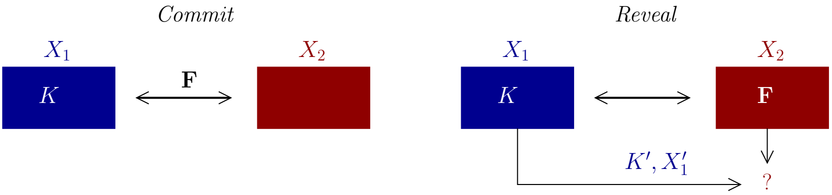
OT Capacity (for IID observations):

Maximum rate (l/n) of OT length (with $\delta_{1n}, \delta_{2n} \to 0$)

$$C_\epsilon(X_1, X_2) \leq \min\{I(X_1 \wedge X_2), H(X_1 \mid X_2)\}$$

"Strong" version of the Ahlswede-Csiszár upper bound

3. Information Theoretic Bit Commitment



Party 2 constructs a test T for the hypothesis: "Secret is k"

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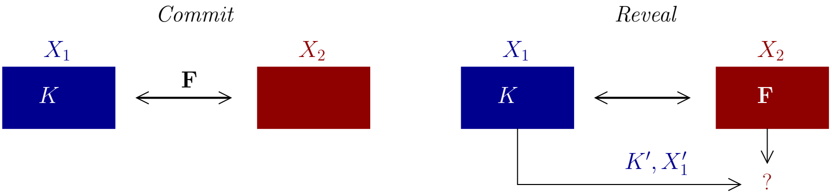

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Efficiency of reduction of BC to OT

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Improves a bound of [Ranellucci et. al. 11]

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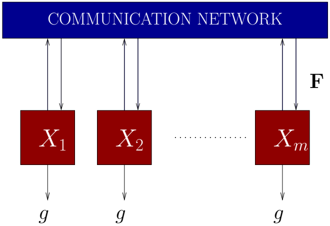
[Nascimento-Winters-Imai 03] BC capacity $C = H(X_1 \mid X_2)$

Strong converse for BC capacity

$$C_{\epsilon,\delta_1,\delta_2}(X_1, X_2) \leq H(X_1 \mid X_2), \quad \epsilon + \delta_1 + \delta_2 < 1$$

4. Secure Computing with Trusted Parties

Parties are trusted, the communication channel is not



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Characterization of securely computable functions

[Tyagi-Gupta-Narayan '11] IID case with $Z = \emptyset$

A function g is secure computable (asymptotically) iff

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A single-shot necessary condition

Theorem

If a function g is (ϵ, δ)-secure computable, then

$$H_{\min}^{\xi}(\mathrm{P}_G) \lesssim \frac{-1}{|\pi| - 1} \log \beta_{\epsilon+\delta+2\xi}\big(\mathrm{P}_{X_{\mathcal{M}}Z}, \mathrm{Q}_{X_{\mathcal{M}}Z}\big),$$

where

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How close do efficient schemes come to these performance bounds??